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Cartan's chains and Lorentz geometry

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Abstract. *4-dimensional Lorentzian geometries admitting a shearfree optical geometry are considered. Cartan's chains defined on a 3-dimensional CR manifold are lifted to null curves. It is proved that all of them are geodesic iff the Weyl tensor is of Petrov type N and the metric admits a twisting null conformal-Killing vector field. Such metrics are shown to be Fefferman's metrics. In another case, there exist at most two congruences of null geodesic chains.*

1. INTRODUCTION

Cauchy-Riemann (CR) geometry appeared in mathematics as a structure characterizing real, $2n-1$ dimensional hypersurfaces in $\mathbb{C}^n$. This definition has subsequently been generalised to deal with an arbitrary odd-dimensional real manifold. Cartan [3] and Chern & Moser [4] defined a family of curves on a CR manifold, which were called chains. They play a similar role to that played by geodesics in the case of Riemann geometry. CR structures find applications in gravitation theory, while Lorentzian metrics are used in the study of CR structures.

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Optical geometry was introduced in gravitation theory by A. Trautman [2] as the simplest structure required for the study of null Maxwell and Yang-Mills fields. If the so-called shear of an optical geometry vanishes, then the optical geometry fixes in a unique and one-to-one manner a certain 3-dim CR geometry. The class of 4-dimensional Lorentzian metrics associated with a shearfree optical geometry (and thus with a corresponding CR-structure) plays an important role in the theory of exact solutions of Einstein's equation [11].

On ther other hand, C. Fefferman [7] (see also [6]) associated a conformal class of 2n-dimensional Lorentzian metrics with each pseudoconvex 2n − 1 dimensional CR geometry (the case of $n = 2$ will be of interest in the sequel). Cartan chains are projections of null geodesics of Fefferman metrics. K. Koch [8] studies a wider class of metrics, null geodesics and their projections on CR manifolds. The properties of curves so obtained are compared to the properties of chains using the example of the Gödel metric in her work.

In the present work, all 4-dimensional metrics connected through shearfree optical geometry with 3-dimensional CR structures are considered. Fefferman and Koch metrics are particular elements of the Trautman class studied here.

In Section 1 the descriptions of optical and CR geometries are presented.

Section 2 reviews the definitions of Cartan-Chern connection and chains.

Section 3 is devoted to the study of the question of when a projection of a null geodesic is a chain. This involves lifting the Cartan-Chern connection to the 4-dimensional spacetime endowed with a Lorentzian metric. Then, making use of this lifted connection, a certain conformally invariant exact 2-form $d\epsilon$ is defined, whose vanishing implies that the metric is of the Fefferman class. It is proved that if the projection of any null geodesic is a chain, then the metric is a Fefferman metric (this is the converse to the well-known theorem [6]). In all other cases, an algebraic equation is derived which singles out at most two null directions at each point. If a null geodesic which projects to a chain passes through a given point, then this geodesic must be tangent to one of these null directions.

In Section 4, the class of metrics admitting a null conformal-Killing vector field is studied in detail. The connection between the form $d\epsilon$ defined in Section 3 and the algebraic properties of the Weyl tensor are studied. It is proved that the necessary and sufficient condition for the null directions defined in Section 3 not to exist is that the metric be of Petrov type III and that the degenerate principal direction of the Weyl tensor be tangent to the conformal Killing vector.

G.A.J. Sparling (in an unpublished work) has proved a theorem that specifies which among 2n-dimensional metrics belong to the Fefferman class. In the present work, a somewhat more stringent statement is proved for the 4-dimensional case: a non-conformally-flat metric is a Fefferman metric if and only if it is of Petrov type N and admits a twisting null conformal-Killing vector field.

These considerations are illustrated by the examples of the Gödel and Taub-NUT metrics.

In the paper we will consider a 4-dimensional manifold M with a metric tensor g of the Lorentz signature $(+++-)$ and a distinguished null vector field k ($k \neq 0$, $g(k, k) = 0$).

An optical geometry [1] of the pair (g, k) is defined by a real 1-form λ and a complex-valued 1-form μ such that

i) $\mu(k) = \lambda(k) = 0$ (1.1)

ii) the metric tensor g takes the form

(1.2) $$g = P^2(\mu\bar{\mu} - \lambda\nu)$$

with an arbitrary function P and 1-form ν.

Given an optical geometry (M, λ, μ), the pair (λ, μ) is determined up to a transformation

(1.3) $$\lambda' = a\,\lambda$$

(1.4) $$\mu' = b(\mu + c\lambda)$$

where a is a real function and b, c are complex functions ($a, b \neq 0$).

Henceforth we will assume that an optical geometry is shearfree; that is,

(1.5) $$\mathscr{L}_k\lambda \wedge \lambda = 0$$

(1.6) $$\mathscr{L}_k\mu \wedge \lambda \wedge \mu = 0$$

where $\mathscr{L}$ denotes Lie derivative. Condition (1.5) is equivalent to the geodesity of k (i.e. due to (1.5) each curve tangent to k is geodesic with respect to g). The vector field k is geodesic and its shear vanishes if and only if (1.5), (1.6) hold.

An optical geometry was defined by A. Trautman [2] in connection with the study of null Maxwell and Yang-Mills fields. A gauge potential is described by a $\mathfrak{g}$-valued 1-form A, where $\mathfrak{g}$ denotes some Lie algebra. It is called null if there exists a (necessarily null) vector field k such that the field strength

(1.7) $$F = \mathrm{d}A + \frac{1}{2}[A, A]$$

fulfills

(1.8) $$k \lrcorner (F + i{*}F) = 0.$$

The Hodge dualisation * depends on a metric g; however, $*F$ is only related to the optical geometry of (g, k). The Yang-Mills equations

$$d{*F} + [A, {*F}] = 0 \tag{1.9}$$

imply that the optical geometry must be shearfree.

A 3-dimensional CR structure consists of a 3-dimensional manifold N and an equivalence class of pairs of 1-forms $[(\lambda, \mu)]$ which satisfies the following assumptions:

i) λ is real and μ is complex;

ii) $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$ (1.10)

iii) any two pairs (λ, μ) and (λ', μ') are related by a transformation (1.3 - 1.4) with functions a, b, c defined on N (a is real and b, c are complex).

Suppose

$$M = \mathbb{R} \times N \tag{1.11}$$

and λ, μ are extended to M by the canonical projection

$$\pi : \mathbb{R} \times N \to N. \tag{1.12}$$

The vector field k (given by (1.1)) is tangent to the fibres of π and a pair (λ, μ) defines an optical geometry on M. One easily verifies that this optical geometry is shearfree.

Conversely, any shearfree optical geometry (M, μ', λ') may be locally represented by a cartesian product (1.11) with some 3-dimensional CR structure (N, μ, λ).

2. CARTAN-CHERN CONNECTION AND CHAINS

Because of the important role played by Cartan-Chern connection and chains in this paper, this short section is devoted to their definitions. They are based on the works of E. Cartan [3] and S.S. Chern, J.K. Moser [4]. We do not use here the 8-dimensional principal fibre bundle which is provided by the $SU(2, 1)$ Cartan connection. The $SU(2, 1)$-valued 1-form ω introduced below is defined directly on the manifold N of the CR structure and is related to the Cartan-Chern connection by an appropriate section of the bundle.

We will assume throughout this work that a CR structure (N, μ, λ) is non-degenerate (and pseudoconvex as a consequence); that is,

$$\lambda \wedge d\lambda \neq 0 \tag{2.1}$$

The requirement (2.1) makes the vector field k consider in the previous section twisting (i.e. $k_{[\alpha}k_{\beta';\gamma]} \neq 0$).

By a transformation (1.3) we normalise λ to satisfy

$$\lambda \wedge d\lambda = i\mu \wedge \bar{\mu} \wedge \lambda. \tag{2.2}$$

Let $\omega = (\omega^j_k)_{j,k=0,1,2}$ denote a matrix of 1-forms,

$$\begin{aligned}
&\omega^0_0 = \frac{1}{3}(2\Pi_2 + \bar{\Pi}_2),\ \omega^0_1 = i\,\bar{\Pi}_3,\ \omega^0_2 = -\frac{1}{2}\Pi_4,\\
&\omega^1_0 = \mu,\ \omega^1_1 = \frac{1}{3}(\bar{\Pi}_2 - \Pi_2),\ \omega^1_2 = -\frac{1}{2}\Pi_3,\\
&\omega^2_0 = 2\lambda,\ \omega^2_1 = 2i\bar{\mu},\ \omega^2_2 = -\frac{1}{3}(2\bar{\Pi}_2 + \Pi_2),
\end{aligned} \tag{2.3}$$

with Π_2, Π_3 being complex and Π_4 being a real 1-form. The expression

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \tag{2.4}$$

gives the curvature Ω. The choice of $(\Pi_a)_{a=2,3,4}$ in (2.3) becomes almost unique if we require $\hat{\Omega}$ to be of the form

$$\begin{aligned}
&\Omega^0_0 = 0,\ \Omega^0_1 = i\bar{R}\mu \wedge \lambda,\ \Omega^0_2 = \frac{1}{2}(S\mu + \bar{S}\bar{\mu}) \wedge \lambda\\
&\Omega^1_0 = 0,\ \Omega^1_1 = 0,\ \Omega^1_2 = -\frac{1}{2}R\bar{\mu} \wedge \lambda\\
&\Omega^2_0 = 0,\ \Omega^2_1 = 0,\ \Omega^2_2 = 0.
\end{aligned} \tag{2.5}$$

The complex functions R and S are defined by (2.5). They can be expressed by coefficients (and their derivatives) of Riemannian connection related to a metric (1.2). However, there is no simple way of describing R and S only by Riemann curvature coefficients.

The remaining freedom consists in trasformations

$$\begin{aligned}
&\Pi'_2 = \Pi_2 + \rho\lambda,\\
&\Pi'_3 = \Pi_3 + \rho\mu,\\
&\Pi'_4 = \Pi_4 + d\rho + \rho(\Pi_2 + \bar{\Pi}_2) + \rho^2\lambda,
\end{aligned} \tag{2.6}$$

where ρ is a real function.

Equations (2.5) subordinate Π_a to the choice of a pair (λ, μ) representing a CR structure.

A transformation of (λ, μ) into (λ', μ') leaving the CR structure (N, μ, λ)

invariant is restricted by (2.2) to

$$\lambda' = t^2\lambda, \tag{2.7}$$
$$\mu' = te^{3i\theta}(\mu + \beta\lambda),$$

where t, θ are real ($t \neq 0$) and β is a complex function.

The functions t, θ, β and ρ used in (2.6) may be collected into a matrix $h \in SU(2, 1)$,

$$h^0_0 = te^{i\theta},\ h^0_1 = i\bar{\beta}e^{-2i\theta},\ h^0_2 = -\frac{e^{i\theta}}{4t}\left(\frac{1}{2}\rho + i|\beta|^2\right),$$

$$h^1_0 = 0,\ h^1_1 = e^{-2i\theta},\ h^1_2 = -\frac{e^{i\theta}}{2t}\beta, \tag{2.8}$$

$$h^2_0 = 0,\ h^2_1 = 0,\ h^2_2 = \frac{1}{t}e^{i\theta}.$$

The convenience of using ω becomes more visible if we write the transformation of ω which follows from (2,6, 2.7) as

$$\omega' = h^{-1}\,\omega h + h^{-1}\,\mathrm{d}h. \tag{2.9}$$

Then the curvature goes into

$$\Omega' = h^{-1}\,\Omega h. \tag{2.10}$$

The 1-form ω given by (2.3 - 2.5) we call the Cartan-Chern connection 1-form.

There is a distinguished class of curves in a CR manifold (N, μ, λ) which were called chains by Cartan [3]. After the above preparation, we are ready to quote Chern-Moser definition. Let n be a curve in N and let $\dot{n}(\tau)$ denote the tangent vector to n at the point $n(\tau)$. Suppose

$$\lambda(\dot{n}) \neq 0. \tag{2.11}$$

Choose (λ', μ') such that along n

$$\mu' = \mu - \frac{\mu(\dot{n})}{\lambda(\dot{n})}\lambda \tag{2.12}$$

and find the connection form ω' associated to (λ', μ'). The curve $n(\tau)$ is a chain iff

$$\omega'^0_1(\dot{n}) = 0. \tag{2.13}$$

We will not need in our considerations an explicit expression for (ω^i_j). It can, however, be obtained by using of (2.2) - (2.5). Such a formula was derived in [5] in the case of (N, μ, λ) admitting a nonconstant solution z of the equation

$$\mathrm{d}z \wedge \mu \wedge \lambda = 0. \tag{2.14}$$

To complete the discussion, let us note that the function R defined by (2.5) is transformed by (2.10) to

$$R' = \frac{1}{t^4}\, e^{6i\theta} R, \tag{2.15}$$

and unless $R = 0$, R may be normalised to get

$$R = 1. \tag{2.16}$$

Then a second condition imposed on h, independent on (2.16),

$$Re\,\omega'^0_0 = 0 \tag{2.17}$$

singles out a pair (λ', μ') up to a change of a sign,

$$\mu' = -\mu.$$

On the other hand, if

$$R = 0, \tag{2.18}$$

then also

$$S = 0, \tag{2.19}$$

and (λ, μ) is equivalent (by means of (1.3 nd (1.4)) to the following (λ', μ')

$$\mu' = \mathrm{d}z, \quad \lambda' = -\frac{1}{2}\,(\mathrm{d}u + i\bar{z}\mathrm{d}z - iz\mathrm{d}\bar{z}), \tag{2.20}$$

where (u, x, y) is the appropriate local chard on the CR manifold N and $z = x + iy$.

3. CHAINS IN LORENTZ GEOMETRY

Throughout this whole work M denotes a 4-dimensional oriented manifold with a pseudoriemannian metric tensor g of the Lorentz signature $(+++-)$. Our basic assumption on g is that it admits a twisting, shearfree, geodesic and null (SGN) vector field k. All considerations are local and refer to neighbourhoods of nonsingular points of presented constructions.

As was explained in Section 1, we identify M locally with the Cartesian product

(3.1) $$M = \mathbb{R} \times N$$

where the 3-dimensional manifold N is equipped with a nondegenerate CR structure (λ, μ). The metric g is written as

(3.2) $$g = P^2(\mu\bar{\mu} - \lambda\mu)$$

with a real function P and a real 1-form ν both, defined on M. Considering all possible g is equivalent to considering all possible (N, μ, λ), P, ν in (3.2).

DEFINITION 3.1. A null curve in M is called a *null chain* iff it projects into a chain in N.

We study in this section the following problem: when is a null chain a null geodesic?

The properties of being a null chain and a null geodesic depend on the conformal geometry of g. We begin with two steps: first we extend the Cartan-Chern connection form to M, and second we define a confomally invariant 2-form involving more information about the conformal geometry (the Cartan-Chern connection – even extended – depends only on the optical geometry of (g, k)).

Let vector fields $(e_i)_{i=1,\ldots,4}$ set up a conformally null frame on M; that is e_3, e_4 are real,

(3.3) $$e_1 = \bar{e}_2 \text{ is complex,}$$

and denoting by $(e^i)_{i=1,\ldots,4}$ the dual coframe to (e_i) we can write

(3.4) $$e^1e^2 - e^3e^4 = \phi^2 g$$

where ϕ is a real function on M.

We restrict ourselves to (e^i) fulfilling

(3.5) $$e_4 \parallel k,$$

(3.6) $$de^3 \wedge e^3 = ie^1 \wedge e^2 \wedge e^3.$$

For example,

(3.7) $$e^1 = \bar{\mu},\ e^2 = \mu,\ e^3 = \lambda,\ e^4 = \nu,$$

with (λ, μ) satisfying (2.2).

Any two coframes (e'^i) and (e^i) are related by a transformation

(3.8) $$e'^1 = te^{3i\theta}(e^1 + \beta e^3)$$

$$e'^3 = t^2 e^3$$

$$e'^4 = e^4 + \bar{\beta} e^1 + \beta e^2 + |\beta|^2 e^3,$$

where t, θ are real and β is a complex function. Put t, θ, β and a real function ρ defined on M in to (2.8) and construct a matrix valued function h. Let (e^i) and (e'^i) be as in (3.7), (3.8). We set

$$\hat{\omega}' = h^{-1}\, \pi^* \omega h + h^{-1}\, dh \tag{3.9}$$

where ω is the Cartan-Chern connection form corresponding to (λ, μ) and π is as in (1.12). The prime in (3.9) arises from (e'^1). The above definition leaves the transformation rules (2.9), (2.10) and the form of $\hat{\Omega}$ in (2.5) invariant (with ω, Ω, μ, λ replaced by $\hat{\omega}$, $\hat{\Omega}$, e^1, e^3 respectively and h associated to (3.8)). It follows from (3.9) that after a coframe change (3.8), a 1-form $\hat{\omega}_1^1$ goes into

$$\hat{\omega}'^1_1 = \hat{\omega}_1^1 - 2\, i d\theta + i\bar{\beta} e^1 + i\beta e^2 + i\beta\bar{\beta} e^3. \tag{3.10}$$

Let us define

$$\epsilon = e^4 + i\hat{\omega}_1^1. \tag{3.11}$$

By (3.8) and (3.10), ϵ is transformed to

$$\epsilon' = \epsilon + 2d\theta. \tag{3.12}$$

Thus a 2-form $d\epsilon$ is uniquely defined.

PROPOSITION 3.1. *If g and g' are associated to the same (N, μ, λ) and*

$$d\epsilon = d\epsilon' = 0, \tag{3.13}$$

then the conformal class $[g']$ is related to $[g]$ by a diffeomorphism which leaves the integral curves of k invariant.

Proof. From (3.13), (3.7) g is conformally equivalent to

$$\hat{g} = \mu\bar{\mu} - \lambda(d\varphi - i\hat{\omega}_1^1) \tag{3.14}$$

where $d\varphi = \epsilon$. Let the functions $(x^i)_{i=1,2,3}$ and the function r be coordinates in N and $\mathbb{R}$ respectively. Then (x^i, r) parameterises $\mathbb{R} \times N$ and

$$\lambda = \lambda_i(x^j)\, dx^i, \quad \mu = \mu_i(x^j)\, dx^i$$

$$\hat{\omega}_l^k = \hat{\omega}_{li}^k\, (x^j)\, dx^i.$$

Defining a diffeomorphism

$$\Phi : (x^i, r) \to (x^i, \varphi(x^i, r)),$$

we can write

(3.15) $$g_F = \Phi^{-1} * \tilde{g} = \mu\bar{\mu} - \lambda(\mathrm{d}r - i\hat{\omega}^1_1).$$

Similarly,

$$g_F = \Phi'^{-1} * g'$$ ∎

which finishes the proof.

DEFINITION 3.1. A metric which fullfils (3.13) is called a *Fefferman metric.* This definition agrees with [6] (see also [5]).

THEOREM 3.1. *The following are equivalent:*

i) $\mathrm{d}\epsilon = 0$;

ii) *each null chain is null geodesic*

iii) *each null geodesic is a null chain.*

Proof. Fix a frame (e_i) (and the dual coframe (e^i)) and set

(3.16) $$\frac{1}{2}\,\epsilon_{ij}\, e^i \wedge e^j = \mathrm{d}\epsilon.$$

Consider a null curve $m(\tau)$ in $\mathbb{R} \times N$. We choose a parametrisation such that the tangent to $m(\tau)$ is

(3.17) $$\dot{m}(\tau) = e_3 - \beta(\tau)e_1 - \bar{\beta}(\tau)\, e_2 + \beta\bar{\beta}\, e_4.$$

Then we extend the complex function β to a neighbourhood of $m(\tau)$ and define a new frame

(3.18)
$$\begin{aligned} e'_1 &= e_1 - \bar{\beta}e_4, \\ e'_2 &= e_2 - \beta e_4, \\ e'_3 &= e_3 - \beta e_1 - \bar{\beta}e_2 + \beta\bar{\beta}e_4, \\ e'_4 &= e_4. \end{aligned}$$

According to Definition 3.1. and (2.13), (3.9), it is easy to see that $m(\tau)$ is a null chain iff

(3.19) $$\hat{\omega}'^0_1(e'_3) = 0$$

On the other hand, the geodesic equation for $m(\tau)$ may be written as

(3.20) $$(\mathrm{d}e'^4)\,(e'_3, e'_1) = 0.$$

By (3.11) and (2.5), which is valid for $\hat{\Omega}'$ with μ, λ being replaced by e'^1, e'^3, it is found that

$$\mathrm{d}e'^4 = \mathrm{d}\epsilon + i(e'^1 \wedge \hat{\omega}'^0_1 - e'^2 \wedge \bar{\hat{\omega}}'^0_1). \tag{3.21}$$

Hence (3.20) gives

$$\epsilon_{31} + (\epsilon_{43} + \epsilon_{12})\bar{\beta} + \epsilon_{24}\bar{\beta}^2 - i\hat{\omega}'^0_1(e'_3) = 0 \tag{3.22}$$

along the curve $m(\tau)$.

We observe that (ii), (iii) follow from $\mathrm{d}\epsilon = 0$. In order to show the implications ii) $\rightarrow$ i) and iii) $\rightarrow$ i), we consider all null geodesic chains which pass through a fixed point m. The assumption ii) (or iii)) implies that (3.22) holds at a point m for any complex $\bar{\beta}$ and with vanishing $\hat{\omega}'^0_1(e'_3)$. Thus

$$\epsilon_{ij} = 0$$

in an arbitrary point $m \in \mathbb{R} \times N$. This completes the proof. ■

We now turn to study the case

$$\mathrm{d}\epsilon \neq 0. \tag{3.23}$$

Let us go back to equation (3.22). Note that a necessary condition for the existence of a null geodesic chain passing through a point m is the existence of a complex solution $\beta(m)$ of the equation

$$\epsilon_{31} + (\epsilon_{43} + \epsilon_{12})\bar{\beta}(m) + \epsilon_{24}\bar{\beta}^2(m) = 0. \tag{3.24}$$

We divide the rest of this section into two parts.

a) Suppose (3.24) has no solution $\bar{\beta}(m)$ in some open set

$$U \subset \mathbb{R} \times N.$$

This means that we have

$$\epsilon_{14} = \epsilon_{24} = \epsilon_{12} = \epsilon_{34} = 0, \tag{3.25}$$
$$\epsilon_{13} \neq 0 \neq \epsilon_{23}.$$

Thus $\mathrm{d}\epsilon$ has the form

$$\mathrm{d}\epsilon = (\Phi\, e^1 + \bar{\Phi}\, e^2) \wedge e^3 \tag{3.26}$$

with a complex function Φ fullfiling the integrability condition

$$\mathrm{d}[(\Phi\, e^1 + \bar{\Phi}\, e^2) \wedge e^3] = 0. \tag{3.27}$$

(3.27) written in the coframe (3.7) reduces to an equation on the function Φ defined on N, (3.28) $$d[(\Phi\mu + \overline{\Phi\mu}) \wedge \lambda] = 0.$$

The same equation appears as one of the Maxwell equations for a null electromagnetic field. Its solubility was discussed by J. Tafel [9], based on the results of H. Jacobowitz and F. Treves [10]. For example, in the case of an analytic (N, μ, λ) eq. (3.28) is soluble, so this class of metrics is not empty.

THEOREM 3.2. *Condition (3.25) is equivalent to the following pair of properties:*
i) k is tangent to a triple principal null direction of the Weyl tensor of g;
ii) k is parallel to a twisting conformal-Killing vector field of g.

The proof of this theorem is presented in the next section (Theorem 4.1). ■

b) Suppose (3.24) admits a solution.

Let the complex function β be given by (3.24) on some open subset $U \subset M$. A congruence of null curves is defined by (tangent to) the vector field

(3.29) $$X = e_3 - \beta e_1 - \overline{\beta} e_2 + \beta\overline{\beta} e_4.$$

As follows from (3.24) there are at most two such congruences.

Conclusion 3.1. Any two of these properties of a curve $m(\tau)$ are equivalent:
i) $m(\tau)$ is a null geodesic;
ii) $m(\tau)$ is a null chain;
iii) one of the vector fields defined by (3.24), (3.29) is tangent to $m(\tau)$.

REMARK 3.1. The requirement that the vector field X be geodesic imposes an extra condition on the conformal class $[g]$. This condition has not been examined yet.

We find now all the congruence of null geodesic chains in the cases of the Gödel and Taub-Nut metrics

Examples

(1) Gödel universe

The formula of the Gödel metric tensor which is most suitable for out purposes was written in [1],

(3.30) $$g = \frac{1}{x^2}\{-2(x\, du - dy)(x\, dr - dy) + dx^2 + dy^2\}.$$

Both vector fields ∂_r and ∂_u are null, geodesic, and shearfree. Let us choose

(3.31) $$k = \partial_r.$$

A CR manifold (N, μ, λ) is represented by the hypersurface $r = 0$ and

(3.32) $$\lambda = 2x\,(x\,\mathrm{d}u - \mathrm{d}y) \qquad \mu = \mathrm{d}z,$$

$$z = x + iy.$$

We construct a coframe in $\mathbb{R} \times N$,

(3.33) $$e^1 = \mu,\ e^2 = \overline{\mu},\ e^3 = \lambda,\ e^4 = \mathrm{d}r - \frac{1}{x}\,\mathrm{d}y.$$

To find all the vector fields X given by (3.24), (3.29), we compute (see (2.3))

(3.34) $$\Pi_2 = \frac{1}{x}\,\mu + \frac{i}{8x^2}\,\lambda,\quad \Pi_3 = \frac{i}{8x^2}\,\mu,$$

$$\epsilon_{14} = \epsilon_{13} = 0$$

We obtain finally the unique vector field

$$X = \partial_u.$$

Let us note that this is exactly the second shearfree geodesic and null vector field admitted by the metric tensor (3.30). The nullness and geodesity of X and Conclusion 3.1 imply that the integral curves of X are the only null chains in this spacetime.

(2) Taub-Nut metric

(3.35) $$g = \frac{1}{2P^2x^2}\,\{\mathrm{d}z\,\mathrm{d}\overline{z} - 2P^2\,|\,\rho\,|^2\,\lambda\,(\mathrm{d}r - U\lambda)\}\ ,$$

where

$$\lambda = \mathrm{d}u + \frac{|\,\rho^0\,|}{2P}\,(iz\,\mathrm{d}\overline{z} - i\overline{z}\,\mathrm{d}z),$$

$$P = 1 + nz\overline{z},\ n = \pm\frac{1}{4}\ \text{or}\ 0,$$

$$x = (r - i\,|\,\rho^0\,|\,)^{-1}.$$

$$U = -2n + |\,\rho\,|^2\,(r\psi + 2n\,|\,\rho^0\,|^2)$$

ψ, ρ^0 – a real constants.

Similarly, as in example (1), we find the unique vector field (up to a scaling factor)

(3.36) $$X = \partial_u + U\partial_r.$$

One can check that X is geodesic (so it is a chain too). Moreover, its shear vanishes.

In the calculations we have used an explicit formula for the Cartan-Chern connection written in [5]. From (3.11) and (3.16) we have derived ϵ_{ij}. Next we have solved (3.24) with respect to β.

Finally we have constructed the vector field X assigned by (3.29).

4. A METRIC ADMITTING A TWISTING NULL CONFORMAL KILLING VECTOR FIELD

We now restrict our attention to the case of the vector field k generating a conformal symmetry of g. But before this, let us recall some properties of unrestricted (even nontwisting) shearfree geodesic and null (SGN) vector fields.

PROPERTY 4.1. [1]. A vector field k is SGN with respect to a metric tensor g if and only if it satisfies

(4.1) $$\mathcal{L}_k g = \varphi g + g(k)\,\xi,$$

where $g(k)$ is a 1-form given by

(4.2) $$g(k)(v) = g(k, v)$$

and φ, ξ are arbitrary.

It is clear that any null conformal-Killing vector field Y

– that is a Y such that

(4.3) $$\mathcal{L}_Y g = \varphi g$$

– is in particular shearfree and geodesis.

Note also that a rescaling of k by a function f via

(4.4) $$\tilde{k} = fk$$

makes (4.1) become

(4.5) $$\mathcal{L}_{\tilde{k}} g = f\varphi g + g(\tilde{k})\,(\xi + \mathrm{d}\,lnf).$$

Therefore the condition

(4.6) $$\mathrm{d}\xi = 0$$

is necessary and sufficient for the existence of a $\tilde{k}$ which is parallel to k and generates a conformal symmetry of g.

PROPERTY 4.2. [1]. If k is SGN with respect to the metric g, then it is tangent to one of the principal null directions of the Weyl tensor.

THEOREM 4.1. *If k is a twisting, null conformal-Killing vector field with respect to g, (e_i) is a frame given by (3.3) - (3.6) and $d\epsilon$ is the 2-form defined by (3.11), then k is tangent to a Weyl tensor principal null direction of multiplicity at least*

a) 1 (always)

b) 2 iff $\epsilon_{14} = \epsilon_{24} = \epsilon_{34} = 0$ (4.7)

c) 3 iff $\epsilon_{14} = \ldots = \ldots = \epsilon_{12} = 0$ (4.8)

d) 4 iff $d\epsilon = 0$ (4.9)

Proof. The multiplicity of a principal null direction k of the Weyl tensor may be defined as follows:

(4.10) k is single $\Longleftrightarrow C_{ijmn} k^i k^m v^j w^n = 0$

(4.11) k is double $\Longleftrightarrow C_{ijmn} k^i k^m v^j = 0$

(4.12) k is triple $\Longleftrightarrow C_{ijmn} k^i v^m = 0$

(4.13) k is quadruple $\Longleftrightarrow C_{ijmn} k^i = 0,$

where v, w run over all vectors orthogonal to k.

Conditions (4.7) - (4.9) are invariant with respect to the admissible transformations (3.8) of the coframe (e^i). Let us fix the coframe (3.7), that is

(4.14) $$e^1 = \mu, \quad e^2 = \overline{\mu}, \quad e^3 = \lambda, \quad e^4 = \nu.$$

Since e_4 is parallel to a conformal-Killing vector, we may use (4.6) with k in (4.1) being replaced by e_4. Thus we obtain

(4.15) $$d(\epsilon_{41} \wedge e^1 + \epsilon_{42} e^2 + \epsilon_{43} e^3) = 0.$$

The vector k, according to Properties 4.1, 4.2 is automatically tangent to the principal null direction; that is, (4.10) holds.

k is double (i.e. (4.11)) iff

(4.16) $$C_{4143} = 0.$$

In the null tetrad (4.14), we get

(4.17) $$C_{4143} = \frac{1}{2} (e_4(\epsilon_{41}) - i\epsilon_{41}).$$

Solving (4.15) - (4.17), we get

(4.18) $$\epsilon_{14} = \epsilon_{24} = \epsilon_{34} = 0,$$

which concludes the proof of b).

The vector k is triple iff

(4.19) $$C_{4143} = C_{4132} = 0.$$

One can compute

$$Im\, C_{4132} = 0$$

(4.20) $$Re\, C_{4132} = -\frac{i}{4}\epsilon_{12},$$

where (4.18) was used (since the 1-form $d\epsilon$ is real its coefficient ϵ_{12} with respect to the basis (e_i) is pure imaginary).

This finishes the proof of c).

Finally, to secure (4.13) it suffices to solve

(4.21) $$C_{4143} = C_{4132} = C_{4331} = 0.$$

When (4.20) and (4.19) are satisfied, then a computation gives

(4.22) $$C_{4331} = -\frac{3}{4}\, i\, \epsilon_{13}.$$

Substituting (4.22) into (4.21) completes the proof of the theorem . ∎

Let us summarize our results concerning Fefferman metrics; that is metrics such that $d\epsilon = 0$. Going back to the expression (3.15), one can see that ∂_r is a Killing vector field of g_F. Therefore, any Fefferman metric admits a conformal-Killing vectorfield. With Theorem 4.1.d), we note that any Fefferman metric is either of Petrov type N or it is conformally flat; i.e.

(4.23) $$C_{ijkl} = 0.$$

It was shown in [5] that a conformally flat metric is related as a Fefferman metric to (N, μ, λ) given by (2.20).

Conclusion 4.1. A 4-dimensional conformally nonflat Lorentz metric is a Fefferman metric if and only if it is of Petrov type N and admits a null twisting vector field, which generates a conformal symmetry.

5. CARTAN-CHERN CONNECTION AND YANG-MILLS EQUATIONS (*)

An optical geometry and a corresponding CR geometry were introduced in

(*) We thank Jacek Tafel for this idea.

[1, 2] as structures adapted to study of nu'' Yang-Mills fields. A Cartan-Chern connection ω extended to the spacetime M (see Section 3) can be understood as an $su(2, 1)$ gauge potential. Moreover, it is also null. It is natural to write for ω the Yang-Mills equations (1.9).

First we find

$$(5\ 1)\qquad *\hat{\Omega} = \begin{bmatrix} 0, & -\overline{R}\mu \wedge \lambda, & \frac{i}{2}(S\mu - \overline{S\mu}) \wedge \lambda \\ 0, & 0 & \frac{i}{2} R\overline{\mu} \wedge \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

Due to the Bianchi identity

$$(5.2)\qquad d\hat{\Omega} + [\hat{\omega}, \hat{\Omega}] = 0,$$

it turns out that the Y-M equations imposed on $\hat{\omega}$ reduce to one real equation defined on N.

$$(5.3)\qquad i\lambda \wedge [d(S\mu - \overline{S\mu}) + R\ \omega_2^0 \wedge \overline{\mu} - \overline{R\omega_2^0} \wedge \mu] = 0.$$

This equation has not yet been investigated. It may be solved explicitly, for example, for CR structures admitting a 3-dimensional symmetry group G_3. Then it has a solution related fo G_3 of the Bianchi type $\mathrm{VII_h}$ (the classification of symmetric CR structures can be find in [12]).

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